$$\left[\frac{b}{2V}\ \frac{\mathrm{d}\phi}{\mathrm{d}t}\right]_{\max} = 0.1 \text{ (full aileron deflection)}$$

According to Eq. (4) the tail load in a full aileron deflection level turn reversal is

$$\frac{Y_v}{mg} = \frac{1}{0.45}\ [2(0.25)^2 + 0.125]\ (0.1) = 0.055{,}55\ldots$$

and the vertical tail size needed for turn reversal at $C_{L_{\max}}$ should be 5-6% of the wing area if the maximum lift coefficient of the vertical tail with deflected rudder is the same as that of the wing. Unflapped single engine light airplanes of the 1930's were in fact provided with vertical tails of this size.

If such an airplane is modified to incorporate an 'STOL conversion'' which doubles or triples the maximum lift coefficient of the wing compared to the vertical tail, Eq. (4) would indicate that the vertical tail size should be doubled or tripled accordingly. Failure to increase the vertical tail size for such an STOL conversion can lead to large excursions in side-slip during "side step" maneuvers on final approach at high C_L with the very real possibility of inadvertent asymmetric stall, loss of control, and crash during the post-stall gyration.

Equation (4) would also indicate that pilots should have difficulty performing low-speed turn reversals with short-tailed aircraft such as sailplanes or "tailless" airplanes, even with heavy rudder "coordination," and I believe they do. The relation apparently provides rationalization for Koppen's rule of thumb: "No airplane with less than a semispan vertical tail arm ever had good stalling properties."

Feedback/Feedforward Matrices for Optimal Following of a Forced Model

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Introduction

DURING the last decade, many papers have considered the design of model reference systems with optimal control theory.[1-8] Most developments apply to the first group of them, that use a *real model* (both free and forced) as a kind of prefilter to the plant. The second group with *implicit model* is not as well documented, and relations for the controller gains are given in general for the free model only. This letter fills the gap by deriving differential equations for the controller matrices when the model also has an input vector.

Problem Statement

Consider the linear constant-coefficients system

$$\mathring{x}_p = A_p x_p + B_p u_p \tag{1}$$

where x_p is the n-vector of states, u_p is the m-vector of inputs, A_p and B_p are matrices of corresponding dimensions. A control u_p is sought such that the derivatives $\mathring{x}_p$ behave in a determined way

$$\mathring{x}_d = A_m x_p + B_m u_m \tag{2}$$

chosen after the equation of a model

$$\mathring{x}_m = A_m x_m + B_m u_m \tag{3}$$

$$\mathring{u}_m = D u_m \tag{4}$$

The objective is to minimize the quadratic performance index

$$I = \tfrac{1}{2}\int_0^T \left[(\mathring{x}_p - \mathring{x}_d)^T Q (\mathring{x}_p - \mathring{x}_d) + u_p^T R u_p \right] \mathrm{d}t \tag{5}$$

The usual methods of optimal control theory can be used to compute the control u of a linear system

$$\mathring{x} = Ax + Bu \tag{6}$$

that minimizes

$$J = \tfrac{1}{2}\int_0^T (\mathring{x}^T Q_o \mathring{x} + u^T R u)\,\mathrm{d}t \tag{7}$$

The Hamiltonian is formed with a vector λ of the costates

$$\begin{aligned} H = {} & \tfrac{1}{2}x^T A^T Q_o A x + x^T A^T Q_o B u \\ & + \tfrac{1}{2}u^T [R + B^T Q_o B]\ u + \lambda^T [Ax + Bu] \end{aligned} \tag{8}$$

Since there are no constraints on u, the extremal control is given by $\partial H/\partial u = 0$ and reads

$$u^* = -\tilde{R}^{-1} B^T [\lambda + Q_o A x] \tag{9}$$

where

$$\tilde{R} = R + B^T Q_o B \tag{10}$$

The extremal Hamiltonian may then be written

$$H^* = \tfrac{1}{2}x^T A^T \tilde{Q}_o A x - \lambda^T \tilde{A} x - \tfrac{1}{2}\,{}^T b \tilde{R}^{-1} B^T \lambda \tag{11}$$

with

$$\tilde{Q}_o = Q_o - Q_o B \tilde{R}^{-1} B^T Q_o \tag{12}$$

$$\tilde{A} = A - B\tilde{R}^{-1} B^T Q_o A \tag{13}$$

In the canonical equations

$$\partial H^*/\partial x = -\mathring{\lambda} = A^T \tilde{Q}_o A x + A^T \lambda \tag{14a}$$

$$\partial H^*/\partial \lambda = \mathring{x} = \tilde{A} x - B\tilde{R}^{-1} B^T \lambda \tag{14b}$$

the costate λ can be eliminated by assuming $\lambda = Px$, where P is a time-varying matrix, given by a Riccati equation

$$\mathring{P} + P\tilde{A} + \tilde{A}^T P - PB\tilde{R}^{-1} B^T P + A^T \tilde{Q}_o A = 0 \tag{15}$$

$$P(T) = 0$$

The control law (9) is then

$$u^* = -\tilde{R}^{-1} B^T [P + Q_o A]\ x \tag{16}$$

Application to the Specific Problem

The problem stated in Ref. 2 can be brought to the form of Ref. 3 by the choice of

$$x = (x_p^T, x_d^T, u_m^T)^T \tag{17a}$$

$$u = u_p \tag{17b}$$

Received March 28, 1975.
Index categories: Aircraft Handling, Stability, and Control; Navigation, Control, and Guidance Theory.
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that leads to the matrices in Eq. (6)

$$A=\begin{bmatrix} A_p & 0 & 0 \\ A_m & 0 & B_m \\ 0 & 0 & D \end{bmatrix} \quad B=\begin{bmatrix} B_p \\ 0 \\ 0 \end{bmatrix} \tag{18}$$

and respectively in Eq. (7):

$$Q_o=\begin{bmatrix} Q & -Q & 0 \\ -Q & Q & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad R \quad \text{unchanged} \tag{19}$$

The tilde matrices are then written:

$$\tilde{R}=R+B_p^TQB_p \tag{20}$$

$$\tilde{Q}_o=\begin{bmatrix} \tilde{Q} & -\tilde{A} & 0 \\ -\tilde{Q} & \tilde{Q} & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{with } \tilde{Q}=Q-QB_p\tilde{R}^{-1}B_p^TQ \tag{21}$$

$$A=\begin{bmatrix} \tilde{A}_p & 0 & B\tilde{R}^{-1}B_p^TQB_m \\ A_m & 0 & B_m \\ 0 & 0 & D \end{bmatrix}$$

$$\text{with } \tilde{A}_p=A_p-B_p\tilde{R}^{-1}B_p^TQ(A_p-A_m) \tag{22}$$

If the matrix $\boldsymbol{P}$ in the Riccati equation is partitioned into 9 submatrices, the relation (15) yields 6 independant equations because of the symmetry of $\boldsymbol{P}$:

$$\mathring{P}_{11}=-P_{11}\tilde{A}_p-P_{12}A_m-\tilde{A}_p^TP_{11}-A_m^TP_{12}$$
$$+P_{11}B_p\tilde{R}^{-1}B_p^TP_{11}-(A_p-A_m)^T\tilde{Q}(A_p-A_m) \tag{23a}$$

$$\mathring{P}_{12}=-\tilde{A}_p^TP_{12}-A_m^TP_{22}+P_{11}B_p\tilde{R}^{-1}B_p^TP_{12} \tag{23b}$$

$$\mathring{P}_{13}=-P_{11}B_p\tilde{R}^{-1}B_p^TQB_m-P_{12}B_m-P_{13}D-\tilde{A}_p^TP_{13}$$
$$-A_m^TP_{23}+P_{11}B_p\tilde{R}^{-1}B_p^TP_{13}+(A_p-A_m)^T\tilde{Q}B_m \tag{23c}$$

$$\mathring{P}_{22}=P_{12}B_p\tilde{R}^{-1}B_p^TP_{12} \tag{23d}$$

$$\mathring{P}_{23}=-P_{12}B_p\tilde{R}^{-1}B_p^TQB_m-P_{22}B_m$$
$$-P_{23}D+P_{12}B_p\tilde{R}^{-1}B_p^TP_{13} \tag{23e}$$

$$\mathring{P}_{33}=-P_{13}B_p\tilde{R}^{-1}B_p^TQB_m-P_{23}B_m-P_{33}D$$
$$-(B_p\tilde{R}^{-1}B_p^TQB_m)^TP_{13}-B_m^TP_{23}$$
$$-D^TP_{33}+P_{13}B_p\tilde{R}^{-1}B_p^TP_{13}-B_m^T\tilde{Q}B_m \tag{23f}$$

From the boundary value $P(T)=0$, it follows successively $P_{12}(t)\equiv 0$, $P_{22}(t)\equiv 0$ and $P_{23}\equiv 0$. Because of the particular from of B in Eq. (18), only the first row of $\boldsymbol{P}$ is involved in the control law (16), that can be written:

$$\boldsymbol{u}_p^*=-\tilde{R}^{-1}B_p^T\{[P_{11}+Q(A_p-A_m)]\boldsymbol{x}_p+$$
$$[P_{13}-QB_m]\boldsymbol{u}_m\} \tag{24}$$

The result $\boldsymbol{P}_{12}(t)\equiv 0$ is consistent with the fact that the vector $\boldsymbol{x}_d$ is fictitious and could not play a role in the real control algorithm. Both other matrices are then solutions of the relations (23a) and (23c) that read after simplification:

$$\mathring{P}_{11}=-P_{11}\tilde{A}_p-\tilde{A}_p^TP_{11}+P_{11}B_p\tilde{R}^{-1}B_p^TP_{11}$$
$$-(A_p-A_m)^T\tilde{Q}(A_p-A_m) \tag{25a}$$

$$\mathring{P}_{13}=-P_{11}B_p\tilde{R}^{-1}B_p^TQB_m-P_{13}D-\tilde{A}_p^TP_{13}$$
$$+P_{11}B_p\tilde{R}^{-1}B_p^TP_{13}+(A_p-A_m)^T\tilde{Q}B_m \tag{25b}$$

with $P_{11}(T)=P_{13}(T)=0$ and the matrices $\tilde{R}$, $\tilde{Q}$ and $\tilde{A}_p$ given in Eqs. (20-22).

Equations (24) and (25) represent the solution to the problem of Eqs. (1-5) for optimal following of a forced model. Previously known equations for the free implicit model[1,5,6] are obtained simultaneously with D, B_m, and P_{13} equal zero.

Conclusions

Kriechbaum and Stineman[9] have recently published similar results, obtained by discrete dynamic programing and letting the time interval approach zero. They also assume that model inputs are slowly varying and can be considered as constant over some time. A detailed comparison shows that their expressions [Eqs. (21-26)] may be obtained from the relations (25) above, with $D\equiv 0$ in Eq. (4). It is thus felt that the present paper, while matching the theoretical line of previous references, offers some more generality than Ref. 9 in allowing time-varying model inputs.

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Wave Structure of Exhausts

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IN a previous Note,[1] a method was suggested whereby shock diamonds could be eliminated by appropriate choice

Received April 7, 1975. The work reported herein was supported by Naval Ordnance Station, Indian Head, Md.

Index categories: Jets, Wakes and Viscid-Inviscid Flow Interation; Airbreathing Propulsion, Subsonic and Supersonic.

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